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THE STABLE VALUE

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Abstract

The paper suggests a new "value" for cooperative games with transferable utilities. The main feature of this value is that it rests on the (both internal and external) stability of the payoffs which members of a deviating coalition can expect to obtain.

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The stable value

Joseph Greenberg

1. Introduction

This paper suggests a new "value" for cooperative games with transferable utilities. The main feature of this value is that it rests on the (both internal and external) stability of the payoffs which members of a deviating coalition can expect to obtain. Loosely speaking, let S be a coalition[1]. An S-Pareto optimal payoff, x, belongs to the set of "reasonable payoffs for S" if and only if there exists no subset T such that the evaluation[2] of the set of "reasonable payoffs for T", yields each of its members a higher payoff than what he gets under x. The stable value (w.r.t. the given evaluation function) is defined to be the evaluation of the set of "reasonable payoffs for the grand coalition, N".

The reader who is familiar with the theory of social situations (Greenberg 1990), will immediately recognize that the logic behind this value is the one which underlies the whole theory. Indeed, if members of a (deviating) coalition T were to consider the <u>best</u> payoff within the set of "reasonable payoffs for T", rather than the evaluation of this set, then the resulting unique "optimistic stable standard of behavior" yields the core correspondence. (See Greenberg 1990, Theorem 6.1.3.) If, in addition, every

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[1] A coalition is a nonempty subset of the set of players.

[2] The way in which the evaluation is done is quite general; see Definition 2.3. For concreteness, assume that a set of payoffs is evaluated according to the expected (w.r.t. some measure) payoff; see Section 4.

coalition - not necessarily subsets - could object to a proposed payoff, then there is a 1-1 correspondence between an optimistic stable standard of behavior (for this "situation") and a von Neumann and Morgenstern solution to the game. (See Greenberg 1990, Theorem 6.2.3.)

We shall, however, be able to define the stable value without (explicitly) using the apparatus and terminology of the theory of social situations.

The paper is organized as follows. The next section provides the formal definition of stability. Perhaps the most important task of the social scientist is to recommend a course of action that will be accepted to the players. In our case, this means recommending a payoff in the set of "reasonable payoffs for the grand coalition". In section 3 we investigate the nonemptiness of this set. Only partial answers are provided: The set of "reasonable payoffs for the grand coalition" is nonempty in every game with a nonempty core, as well as in all two and three-player superadditive games. Whether such is the case with all superadditive games remains an interesting open question (even when cast within "classical cooperative game theory"). Section 4 studies a particularly appealing evaluation function, based on the uniform distribution. We provide a full characterization of the stable value for two and three-person games as well as for all symmetric games. The last section contains a few examples which shed more light on the stable value, and distinguish it from the Shapley value (Shapley 1953). In order not to interrupt the conceptual flow of the paper, all proofs are relegated to the Appendix.

2. Stability

<u>Definition 2.1</u>: An n-person game in characteristic function form with transferable utilities, hereafter ***a game***, is a pair (N,μ) where $N=\{1,2,\ldots,n\}$ is the finite set of players, and μ is the characteristic function which assigns to each coalition S, $S\subset N$, its "worth" - a nonnegative number $\mu(S)\in R_+$.

One common interpretation of the worth of coalition S is that $\mu(S)$ denotes the monetary profits S can generate if and when it forms, and all players have the same, constant, marginal utility of money. For a coalition $S\subset N$, the sets of **S-feasible** and of **S-Pareto** optimal payoffs are given, respectively, by

$$v(S)\equiv\{x\in R^S_+ \mid \Sigma_{i\in S}x^i\leq\mu(S)\} \quad\text{and}\quad v^*(S)\equiv\{x\in R^S_+ \mid \Sigma_{i\in S}x^i=\mu(S)\}.$$

A central question is: Given a game (N,μ), which payoffs will be accepted by ("rational") players? The answer to this question depends on what the players think (know, believe, anticipate) will happen if a payoff is rejected. That is, for each coalition S, $S\subset N$, there is associated a subset, $\sigma(S)$, of $v^*(S)$, which specifies "the set of reasonable payoffs for S, if and when it forms." It is only on the basis of $\sigma(S)$ that members of S can decide whether or not to reject a proposed payoff. A mapping that assigns to each coalition S, $S\subset N$, a subset, $\sigma(S)$, of $v^*(S)$, is called a ***standard of behavior (SB)***. In order for the SB σ to be adopted by "rational" players, we shall require that it be "stable": For all $S\subset N$, $\sigma(S)$ consists of all those, and only those payoffs that will not be rejected by members of S, who are aware of, and believe in the specifications of the SB σ. More specifically, given the SB σ, an S-Pareto optimal payoff x will not be accepted by S if and only if there exists a subset T all of whose members prefer $\sigma(T)$ over x.

The difficulty in formalizing this idea stems from the fact that we need to compare a single payoff, x, with a set of payoffs, $\sigma(T)$. There is no general rule according to which such a comparison can be made. We shall, therefore, allow for any general evaluation of this set, provided that it is common to all members of T. More specifically, let $U \subset v^*(T)$. The <u>evaluation of U</u> is a payoff in $v^*(T)$. (For example, it can be the expectations of U w.r.t. some probability measure; see Section 4.) Since no constraints are imposed on the set U, we cannot require that every set U be evaluated. (For example, U may be empty, open, non-measurable, etc.) In general, therefore, we may want to restrict our attention to a subset of

$$F \equiv \{(S,U) \mid S \subset N \text{ and } U \subset v^*(S)\}.$$

<u>Definition 2.2</u>: Let $D \subset F$. A function φ, which assigns to every $(S,U) \in D$ a payoff, $\varphi(U) \in v^*(S)$, is called an <u>evaluation function</u> (over the domain D).

Given an evaluation function, φ, (over some domain $D \subset F$), we can now formalize the notion of "stable recommendations": The SB σ is stable if the set of payoffs, $\sigma(S)$, which are recommended to members of S contains all those, and only those payoffs that will not be rejected by S, where rejection is based on φ and σ. That is, an S-Pareto optimal payoff x will be rejected by S if and only if there exists T, $T \subset S$, all of whose members prefer the set $\sigma(T)$ (evaluated by φ) over the payoff x. Formally,

<u>Definition 2.3:</u> Let (N,μ) be a game and let φ be an evaluation function over DCF. The SB σ is called ***<u>stable w.r.t. φ</u>*** if it satisfies: For all $S\subset N$, and for all $x\in v^*(S)$,

$x\in v^*(S)\setminus\sigma(S) \Leftrightarrow$ there exists $T\subset S$ s.t. $(T,\sigma(T))\in D$ and $x^i<\varphi^i(\sigma(T))$ for all $i\in T$.

That is, if σ is a stable SB then the following two conditions hold:
(1) **Internal stability**: The SB σ is free of "inner contradictions": If $x\in\sigma(S)$ then there is no subset T all of whose members are better-off [according to φ and σ] if T forms than they currently are. That is, if $x\in\sigma(S)$ then there exists no $T\subset S$ s.t. $(T,\sigma(T))\in D$ and $x^i<\varphi^i(\sigma(T))$ for all $i\in T$. It follows that if players believe in σ, and if they evaluate a set of payoffs according to the function φ, then no payoff that belongs to $\sigma(S)$ will be rejected by any subset of S. Thus, the SB σ is (internally) consistent.
2. **External stability**: The SB σ accounts for all those payoffs that it rules out: Every S-Pareto optimal payoff that is excluded from $\sigma(S)$ would be rejected by some coalition $T\subset S$ whose members expect to be better-off by forming T, believing that the set of payoffs that might obtain once T forms is $\sigma(T)$. That is, if $x\in v^*(S)\setminus\sigma(S)$ then there exists $T\subset S$ s.t. $(T,\sigma(T))\in D$ and $x^i<\varphi^i(\sigma(T))$ for all $i\in T$. Thus, the SB σ cannot arbitrarily label a payoff as "unreasonable".

The following two claims assert that the two requirements of internal and external stability are satisfied by exactly one SB. Moreover, "the set of reasonable outcomes" for any $S\subset N$, according to this unique stable SB, has the following appealing topological property: It is either empty or else it is a finite union of compact and convex sets.

<u>Claim 2.4:</u> Let (N,μ) be a game and let φ be an evaluation function over the domain DCF. Then, there exists a unique SB σ which is stable w.r.t. φ.

<u>Proof:</u> See Appendix.

<u>Claim 2.5:</u> Let (N,μ) be a game, let φ be an evaluation function over the domain DCF, and let σ be the stable SB for (N,μ). Then, for all $S\subseteq N$, $\sigma(S)$ is either empty or else it is a finite union of compact and convex sets.

<u>Proof:</u> See Appendix.

Since the evaluation of $(S,U)\in D$ equals to $\varphi(U)\in v^*(S)$, it seems natural to define <u>player i's evaluation of the game</u> (N,μ) to be equal to his evaluation (w.r.t. φ) of the set of "reasonable payoffs for the grand coalition", provided this evaluation is well-defined, i.e., that $(N,\sigma(N))\in D$. We therefore have the following

<u>Definition 2.6:</u> Let (N,μ) be a game, and let φ be an evaluation function over the domain D such that $(N,\sigma(N))\in D$, where σ is the stable SB w.r.t. φ. The **<u>stable value (w.r.t. φ)</u>** of (N,μ) is $\varphi(\sigma(N))$.

Note that the stable value (w.r.t. φ) need not itself belong to the set of "reasonable payoffs for the grand coalition". That is, it is possible that $\varphi(\sigma(N))\notin\sigma(N)$.[1] (See Examples 5.1 and 5.2).

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1 Just as the expectations from throwing a die is 3.5 - a number that appears on no face of the die.

3. An Open Question

As mentioned in the Introduction, perhaps the most important task of the social scientist is to recommend a course of action that will be accepted to the players. In our case, this means recommending a payoff in the set of "reasonable payoffs for the grand coalition", $\sigma(N)$. This section investigates the nonemptiness of this set. Recall

<u>Definition 3.1:</u> Let (N,μ) be a game. The *<u>core</u>* of the game (N,μ), denoted Core(N,μ), is given by

$$\mathrm{Core}(N,\mu) \equiv \{x \in v^*(N) \mid \text{ for all } S \subset N, \quad \Sigma_{i \in S} x^i \geq \mu(S)\}.$$

<u>Claim 3.2:</u> Let (N,μ) be a game, let φ be an evaluation function over DCF. Then, Core$(N,\mu) \subset \sigma(N)$, where σ is the unique stable SB (w.r.t. φ). In particular, if Core$(N,\mu) \neq \emptyset$ then $\sigma(N) \neq \emptyset$.

<u>Proof:</u> See Appendix.

Claim 3.2 implies that for "balanced games" (see Bondareva 1962, and Shapley 1967), the stable SB (w.r.t. any φ), σ, contains "reasonable outcomes", i.e., $\sigma(N) \neq \emptyset$. However, such is clearly not the case with all games. For example, consider a two-person game where $\mu(1)+\mu(2)>\mu(N)$. Then, for all $x \in v^*(N)$ there exists $i \in N$ such that $x^i<\mu(i)$. It follows that $\sigma(N)=\emptyset$ for all evaluation functions φ which satisfy the very mild and natural condition: $\varphi(v^*(i))=\mu(i)$, $i \in N$. This observation raises the question whether superadditivity is a sufficient condition for $\sigma(N)$ to be nonempty in an arbitrary n-person game,[1] where,

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[1] For $n \cdot 3$, superadditivity is not a necessary condition for $\sigma(N)$ to be nonempty.

<u>Definition 3.4:</u> The game (N,μ) is called <u>superadditive</u> if for all $S,T\subset N$ with $S\cap T=\emptyset$, we have that $\mu(S)+\mu(T)\leq\mu(S\cup T)$.

Whether or not the domain of games which admit "reasonable outcomes" includes all superadditive games remains an <u>open question</u>. While the motivation stems from our stability criterion, this open question can be posed within the "classical paradigm" of game theory:

Q: Let (N,μ) be a superadditive game. For each $S\subset N$, $S\neq N$, let $\xi(S)\in v(S)$. Does there exist a payoff $y\in v(N)$, [hence in $v^*(N)$], such that there is no S with $\xi^i(S)>y^i$ for all $i\in S$?

To realize that a game (N,μ) admits "reasonable outcomes" (for every evaluation function φ) if and only if Q is satisfied, (i.e., answered in the affirmative), define:

$\xi(S)\equiv\varphi(\sigma(S))$ if $(S,\sigma(S))\in D$, and $\xi(S)\equiv 0$ otherwise.

<u>Remark 3.5:</u> When investigating Q, we can, w.l.o.g., consider only **0-normalized superadditive games**, i.e., games (N,μ) where $\mu(i)=0$ for all $i\in N$. Indeed, consider any superadditive game, (N,μ). Define the 0-normalized game (N,υ) as follows: For all $S\subset N$, $\upsilon(S)\equiv\mu(S)-\Sigma_{i\in S}\,\mu(i)$. Then, it is easily verified that (N,υ) is superadditive, and moreover, (N,μ) satisfies Q if and only if (N,υ) satisfies it.

By Claim 3.2, the open question Q is answered in the affirmative whenever (N,μ) admits a nonempty core. The following two claims assert that Q is satisfied also in every two and three-person superadditive game.

<u>Claim 3.6:</u> Every two-person superadditive game satisfies Q.
<u>Proof:</u> See Appendix.

<u>Claim 3.7:</u> Every three-person superadditive game satisfies Q.
<u>Proof:</u> See Appendix.

4. Uniform distribution

A particularly appealing evaluation function is derived from the often employed assumption of "insufficient reason": In the absence of an exogenously given probability distribution over a set of payoffs, U, players assume that all such payoffs are <u>equally likely</u>. Thus, a player's evaluation of a set of payoffs U equals his expected payoff from this set, when the expectations are taken using the (multidimensional Lebesgue measure) uniform distribution over U, (assuming this distribution can be defined over U). It seems to me that, at least within the framework of games with transferable utilities, such an evaluation is quite reasonable. Define, therefore,

$$D^* \equiv \{(S,U) \mid S \subset N, \text{ and } U \subset v^*(S) \text{ s.t. the uniform distribution can be defined over } U\}.$$

For $(S,U) \varepsilon D^*$, let $E(U)$ denote the S-dimensional payoff vector which equals the expectations of U, using the uniform distribution. The SB σ is called <u>***stable***</u> if it is stable w.r.t. the evaluation function E (defined over the domain D^*).

By Claim 2.4, there exists a unique stable SB, σ. For a game (N,μ) such that $(N,\sigma(N))\varepsilon D^*$, the **stable value** is defined to be $E(\sigma(N))$. The following claim shows that the stable value is well-defined whenever $\sigma(N)\neq\emptyset$, (re-enforcing the interest in the open question Q).

Claim 4.1: Let σ be the stable SB for the game (N,μ). Then, for all $S\subset N$, $\sigma(S)\neq\emptyset$ implies $(S,\sigma(S))\varepsilon D^*$, i.e., $E(\sigma(S))$ is well-defined.

Proof: See Appendix.

By Claim 4.1, we have that a SB σ is stable if for all $S\subset N$,

$$x\varepsilon v^*(S)\setminus\sigma(S) \Leftrightarrow \text{there exists } T\subset S \text{ s.t. } \sigma(T))\neq\emptyset$$
$$\text{and } x^i<E^i(\sigma(T)) \text{ for all } i\varepsilon T.$$

It is straightforward to verify that the stable value shares the following three properties:

Pareto optimality: The stable value distributes the entire worth of a coalition. (Thus, E is an evaluation function.) Formally, for all $S\subset N$ with $\sigma(S)\neq\emptyset$, $\Sigma_{i\varepsilon S}E^i(\sigma(S))=\mu(S)$.

Symmetry: Symmetric (substitute) players are assigned the same expected payoff. Formally, let $S\subset N$ and $i,j\varepsilon S$ be such that for all $T\subset S$, if $\{i,j\}\cap T=\emptyset$ then $\mu(T\cup\{i\})=\mu(T\cup\{j\})$. Then, $x\varepsilon\sigma(S)$ if and only if $y\varepsilon\sigma(S)$ where $y^i\equiv x^j$, $y^j\equiv x^i$, and for $t\varepsilon S\setminus\{i,j\}$, $y^t=x^t$. If, in addition, $\sigma(S)\neq\emptyset$ then $E^i(\sigma(S))=E^j(\sigma(S))$.

Strategic equivalence: The stable value is invariant under positive linear transformations of the game. Formally, Let (N,μ) and (N,ω) be two games such that there exists a positive number α, and a vector $b \in R^N$ such that for all $S \subset N$, $\omega(S) = \alpha\mu(S) + \Sigma_{i \in S} b^i$. Then, $x \in \sigma^{\mu}(S)$ if and only if $(\alpha x + b^S) \in \sigma^{\omega}(S)$, where σ^{μ} and σ^{ω} are the unique stable SBs for, respectively, (N,μ) and (N,ω). In particular, if $\sigma^{\mu}(S) \neq \emptyset$ (hence, $\sigma^{\omega}(S) \neq \emptyset$) then $E^i(\sigma^{\omega}(S)) = \alpha E^i(\sigma^{\mu}(S)) + b^i$, $i \in S$.

The symmetry property together with Claim 3.6 yield

<u>Corollary 4.2:</u> Let (N,μ) be a two-person superadditive game. Then, the unique stable SB is nonempty-valued, and is given by: For $i \in N$, $\sigma(i) = v^*(i) = \{\mu(i)\}$, and $\sigma(N) = \mathrm{Core}(N,\mu)$. In particular, the stable value, $E(\sigma(N))$, for this game is given by: $E^1(\sigma(N)) = \mu(1) + \delta/2$, $E^2(\sigma(N)) = \mu(2) + \delta/2$, where δ is the surplus from forming N, i.e., $\delta = [\mu(N) - \mu(1) - \mu(2)]$.

That is, the two players split evenly the surplus δ, which, in view of the superadditivity of the game is nonnegative. By Claim 3.7, the stable SB for a three-person superadditive game is nonempty-valued. The following explicit formula for the stable value of such games was derived by Dave Furth and Pieter Kop Jansen.[1] In view of the fact that σ is invariant under positive linear transformations, it is sufficient to consider 0-1 normalized games. That is, the characteristic function μ satisfies:

$$\mu(i)=0 \text{ for all } i \in N \ ; \ \mu(2,3)=\alpha \ ; \ \mu(1,3)=\beta \ ;$$
$$\mu(1,2)=\gamma \ ; \ \mu(N)=1, \text{ where, w.l.o.g., } 1 \geq \alpha \geq \beta \geq \gamma \geq 0.$$

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[1] Pieter Kop Jansen provided also a computer program that calculates the stable value for all three person games.

Denote the two scalars:

$$a \equiv \mathrm{Max}\{0,\ \alpha/_2+\beta-1\}\ ;\ b \equiv \mathrm{Max}\{0,\ \alpha/_2+\gamma-1\},$$

and the five vectors in R^3:

$$\xi \equiv (1-\alpha/_2,\ \alpha/_4,\ \alpha/_4)\ ;\ \eta \equiv (\beta/_4,\ 1-\beta/_2,\ \beta/_4)\ ;$$

$$\zeta \equiv (\gamma/_4,\ \gamma/_4,\ 1-\gamma/_2)\ ;\ \rho \equiv (1-\alpha/_2+\beta/_2,\ 1-\beta+\alpha,\ 1-\alpha/_2+\beta/_2)\ ;$$

$$\tau \equiv (1-\alpha/_2+\gamma/_2,\ 1-\alpha/_2+\gamma/_2,\ 1-\gamma+\alpha).$$

Finally, define

$$\theta \equiv 1+a^2+b^2-(\alpha^2+\beta^2+\gamma^2)/_2.$$

The stable value for the game (N,μ) is given by: For $\theta=0$

$$E(\sigma(N))=(^1/_3,{}^1/_3,{}^1/_3),$$

and for $\theta\neq 0$,

$$E(\sigma(N))=[(^1/_3,{}^1/_3,{}^1/_3)-{}^1/_2\alpha^2\xi-{}^1/_2\beta^2\eta-{}^1/_2\gamma^2\zeta+{}^1/_3\rho a^2+{}^1/_3\tau b^2]/\theta.$$

As the following claim asserts, n-person superadditive and symmetric games provide another class of games for which the stable value is well-defined. (Recall that (N,μ) is a symmetric game if $\mu(S)=\mu(T)$ whenever[1] $|S|=|T|$).

<u>Claim 4.3:</u> If (N,μ) is a superadditive and symmetric game then the stable SB σ is nonempty-valued. Moreover, the stable value is given by: $E(\sigma(N))=(\mu(N)/n,\mu(N)/n,\dots,\mu(N)/n)$.

<u>Proof:</u> See Appendix.

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1 For a finite set B, $|B|$ denotes the cardinality of (number of elements in) B.

5. Examples

The following examples shed some more light on the proposed stable value.

<u>Example 5.1:</u> Consider the well-known three-person majority rule game (N,μ), where every two or three players can share a dollar. That is,

$$N=\{1,2,3\} \ ; \ \text{for } i\in N\text{: } \mu(i)=0 \ ; \ \text{and for } S\subset N,\ |S|>1\text{: } \mu(S)=1.$$

By Corollary 4.2, we have that $E(\sigma(i,j))=(1/2,1/2)$. It follows that

$$\sigma(N)=\{(1/2,1/2,0),(1/2,0,1/2),(0,1/2,1/2)\}.$$

That is, the only payoffs the players can reasonably expect to get (i.e., those payoffs that will not be rejected by any coalition) are those obtained by two of the players dividing the dollar equally among themselves. These recommendations seem, at least to me, to be very appealing.[1] Note that, as asserted in Claim 3.7, the stable value of this game is $E(\sigma(N))=(1/3,1/3,1/3)$, and it is not, itself, a member of $\sigma(N)$.

It is both appropriate and instructive to compare the stable value with the well-known Shapley value (Shapley 1953). The following example demonstrates that these are two distinct concepts. In particular, the stable value does not satisfy the "dummy axiom". Moreover, and again unlike the Shapley value, the stable value need not belong to the core of a "convex game" (Shapley 1971). (Recall that (N,μ) is **convex** if for all $S,T\subset N$, we have that $\mu(S)+\mu(T)\leq\mu(S\cup T)+\mu(S\cap T)$.)

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[1] In contrast, as is well-known, the core of this game is empty, and there are a continuum of vN&M solutions. There is, however, a unique symmetric vN&M solution; it coincides with $\sigma(N)$.

<u>Example 5.2:</u> Consider the convex game (N,μ) where

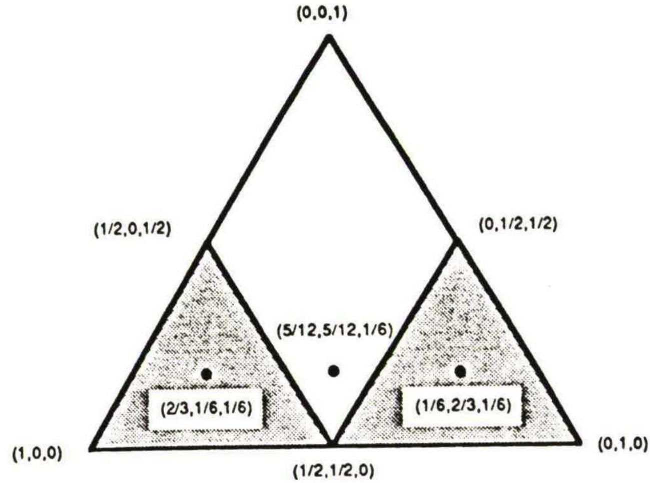
$$N=\{1,2,3\}; \quad \mu(N)=\mu(1,2)=1 \quad ; \text{ and } \quad \mu(S)=0, \quad \text{otherwise.}$$

By Corollary 4.2, $E(\sigma(1,2))=(1/2,1/2)$. Thus,

$$\sigma(N)=\{x\epsilon v^*(N) \mid x_1 \geq 1/2\} \cup \{x \mid x_2 \geq 1/2\}.$$

Hence, the stable value is $E(\sigma(N))=(5/12,5/12,1/6)$. (See Figure 5.2.) In contrast, the Shapley value for this game is the payoff $(1/2,1/2,0)$. Being a convex game, the Shapley value belongs to $\mathrm{Core}(N,\mu)=\{x\epsilon v^*(N) \mid x_1+x_2=1$ and $x_3=0\}$, while $E(\sigma(N))$ does not.

Note that player 3 is a "dummy player" - his marginal contribution is 0 in all coalitions to which he belongs. Nevertheless, the stable value assigns player 3 the (expected) payoff of 1/6. This might perhaps seem implausible. I would like to argue otherwise. Assume that the players consider to divide the dollar among themselves according the the payoff vector $y=(0.9,0,0.1)$. The conventional argument against this payoff is that player 2 can suggest to player 1 to form the coalition $\{1,2\}$, and to equally divide between them the 0.1 dollars that player 3 currently gets. <u>If</u> player 2 can <u>commit</u> himself to this division once "3 leaves the scene", then it does indeed seem most reasonable that y cannot be the final payoff. (Indeed, y is not in the core of the game.) But in the absence of such a commitment, I very much doubt that player 1 will be willing to form coalition $\{1,2\}$ and risk the distinct possibility that player 2, after player 3's departure, will renege. In fact, player 2 might even demand 0.9 dollars and player 1 might have no choice but to accept this offer, because his alternative, once 3 is gone, is to get 0. All player 1 can expect to get once he remains only with player 2, is 0.5 dollars, as asserted in Corollary 4.2. Thus, player 3 might play an important "distributional" (albeit not "productive") role, which



σ (N) is the shaded area

Figure 5.2

accounts for his expected payoff of 1/6, despite the fact that he is a dummy player.

The last example, due to Dave Furth and Roald Ramer, generalizes the above two, and, more importantly, demonstrates that the stable value might exhibit "discontinuities".

<u>Example 5.3:</u> Consider the game (N,μ^{α}) where α, $0\leq\alpha\leq 1$, is a parameter that satisfies

$$N=\{1,2,3\} \;;\; \mu^{\alpha}(i)=0 \text{ for all } i\varepsilon N \;;$$

$$\mu^{\alpha}(1,2)=1 \;;\; \mu^{\alpha}(2,3)=\mu^{\alpha}(1,3)=\alpha \;;\; \mu^{\alpha}(N)=1.$$

Thus, when $\alpha=1$ we get Example 5.1, and $\alpha=0$ yields Example 5.2. Let σ^{α} be the unique stable SB for (N,μ^{α}). It can be shown (see formula for the stable value of a three-person game given in the previous section) that for $1/2<\alpha<1$,

$$E(\sigma^{\alpha}(N))=((4-\alpha)/8,(4-\alpha)/8,\alpha/4).$$

Hence, $\lim_{\alpha\rightarrow 1}E(\sigma^{\alpha}(N))\equiv(3/8,3/8,1/4)$. But, as verified in Example 5.1, for $\alpha=1$ we have that

$$E(\sigma^{1}(N))=(1/3,1/3,1/3)\neq(3/8,3/8,1/4).$$

A possible explanation for this discontinuity (in α) is that there is a qualitative difference between being "exactly as powerful as" to being "nearly as powerful as". And, it is this difference that accounts for the fact that in any game (N,μ^{α}) with $\alpha<1$, the stable value assigns players 1 and 2, a payoff which exceeds that which it assigns to player 3.

Appendix

<u>Claim 2.4:</u> Let (N,μ) be a game and let φ be an evaluation function over the domain DCF. Then, there exists a unique SB σ which is stable w.r.t. φ.

<u>Proof:</u> By induction on the cardinality of N. For $n=1$, i.e., $N=\{i\}$, $\sigma(N)=v^*(N)=\{\mu(i)\}$, and thus the assertion holds.

Assume the validity of the claim for all games with less than n players, and let N consist of n players. Then, by the induction hypothesis, for all $S\subset N$, $S\neq N$, there exists a unique stable SB, σ_S, for the (sub)game (S,μ_S), where for all $T\subset S$, $\mu_S(T)\equiv\mu(T)$. Define the SB σ as follows:

$$\sigma(S)\equiv\sigma_S(S), \quad \text{if} \quad S\subset N,\ S\neq N,$$

$$\sigma(N) \equiv \{x\in v^*(N) \mid \text{ there is no } S\subset N,\ S\neq N, \text{ s.t. } (S,\sigma(S))\in D \text{ and } \varphi^i(\sigma_S(S))>x^i \text{ for all } i\in S\}.$$

It is easy to verify that σ is stable for (N,μ). Moreover, the induction hypothesis implies that for all $S\subset N$, $S\neq N$, $\sigma_S(S)$ is unique. Since $\sigma(N)$ is defined solely on the basis of $\sigma_S(S)$, $S\neq N$, it follows that σ is the unique stable SB for (N,μ).

Q.E.D.

<u>Claim 2.5:</u> Let (N,μ) be a game, let φ be an evaluation function over the domain DCF, and let σ be the stable SB for (N,μ). Then, for all $S\subset N$, $\sigma(S)$ is either empty or else it is a finite union of compact and convex sets.

<u>Proof:</u> The proof is by induction on the number of players in the coalition S. For $|S|=1$, $\sigma(S)=v^*(S)$. Assume the validity of the claim for all coalitions with less than p players, and let S consist of p players. Denote:

$$\Theta\equiv\{T\subset S \mid T\neq S,\ \sigma(T)\neq\emptyset, \text{ and } (T,\sigma(T))\in D\}.$$

Consider $T\varepsilon\Theta$. By the induction hypothesis, $\sigma(T)$ is a finite union of compact and convex sets. For $i\varepsilon T$ denote the set

$$A^i(T)\equiv\{x\varepsilon v^*(S) \mid \varphi^i(\sigma(T))\leq x^i\}.$$

Then, $A^i(T)$ is a closed, hence compact, and convex subset of $v^*(S)$. Thus, the set $A(T)\equiv\cup_{i\varepsilon T}A^i(T)$ is a finite union of compact and convex subsets of $v^*(S)$. Now, $\sigma(S)=\cap_{T\varepsilon\Theta}A(T)$. Hence, $\sigma(S)$ is either empty or else it is a finite union of compact and convex sets (of the form $\cap_k A^{i(k)}(T_k)$ where $T_k\varepsilon\Theta$, $i(k)\varepsilon T_k$).

Q.E.D.

__Claim 3.2:__ Let (N,μ) be a game and let φ be an evaluation function over DCF. Then, $\mathrm{Core}(N,\mu)\subset\sigma(N)$, where σ is the unique stable SB (w.r.t. φ). In particular, if $\mathrm{Core}(N,\mu)\neq\emptyset$ then $\sigma(N)\neq\emptyset$.

__Proof:__ Let $x\varepsilon\mathrm{Core}(N,\mu)$. Then, there are no $S\subset N$ and $y\varepsilon v^*(S)$ such that $y^i>x^i$ for all $i\varepsilon S$. It follows that there is no $S\subset N$ such that $\varphi^i(\sigma(S))>x^i$ for all $i\varepsilon S$. Hence, $x\varepsilon\sigma(N)$.

Q.E.D.

__Claim 3.6:__ Every two-person superadditive game satisfies Q.

__Proof:__ Such games have a nonempty core. The validity of the claim follows, therefore, from Claim 3.2.

Q.E.D.

__Claim 3.7:__ Every three-person superadditive game satisfies Q.

<u>Proof:</u> By remark 3.5, assume, w.l.o.g., that (N,μ) is 0-normalized. Then, we have to consider only the 3 two-player coalitions. W.l.o.g., let player 1 and coalition $T=\{1,2\}$ be such that

$$\xi^1(T)\geq\xi^i(S) \text{ for all } i \text{ and all } S\subset N,\ |S|=2.$$

Define the payoff y as follows: $y^1\equiv0$, $y^2\equiv\xi^2(T)$, and $y^3\equiv\xi^1(T)$. By superadditivity, $\mu(1,2)\leq\mu(N)$, hence, $y\in v(N)$. Now, every coalition $S\subset N$, $|S|=2$, $S\neq T$, contains player 3. By the choice of player 1 and the coalition T, $y^3\geq\xi^3(S)$ for all such S, and hence, no coalition taht contains player 3 can block y. And, since $y^2=\xi^2(T)$, T can not block y, either.

Q.E.D.

<u>Claim 4.1:</u> Let σ be the stable SB for the game (N,μ). Then, for all $S\subset N$, $\sigma(S)\neq\emptyset$ implies $(S,\sigma(S))\in D^*$, i.e., $E(\sigma(S))$ is well-defined.

<u>Proof:</u> By Claim 2.5, $\sigma(S)$ is either empty or else it is a finite union of compact and convex sets, for all $S\subset N$. Since the relative interior of a convex set is nonempty, it follows that a compact and convex set has a positive finite Lebesgue measure in the appropriate dimension (i.e., that of the smallest linear manifold containing it), and in higher dimensions, its Lebesgue measure is 0. Therefore, the uniform distribution can be defined over a finite union of compact and convex sets. Thus, $E(\sigma(S))$ is well-defined for all $S\subset N$ for which $\sigma(S)\neq\emptyset$.

Q.E.D.

<u>Corollary 4.2:</u> Let (N,μ) be a two-person superadditive game. Then, the unique stable SB is nonempty-valued, and is given by: For $i\in N$, $\sigma(i)=v^*(i)=\{\mu(i)\}$, and $\sigma(N)=\text{Core}(N,\mu)$. In particular, the stable value, $E(\sigma(N))$,

for this game is given by: $E^1(\sigma(N))=\mu(1)+\delta/2$, $E^2(\sigma(N))=\mu(2)+\delta/2$, where δ is the surplus from forming N, i.e., $\delta=[\mu(N)-\mu(1)-\mu(2)]$.

<u>Proof:</u> By Claim 3.2, $\sigma(N)\supset Core(N,\mu)$. Now, since $n=2$, only single players can block a Pareto optimal payoff. That is, we have that $x\epsilon v^*(N)\setminus Core(N,\mu)$ $\Leftrightarrow$ there exists $i\epsilon N$ such that $x^i<\mu(i)$.

Q.E.D.

<u>Claim 4.3:</u> If (N,μ) is a symmetric and superadditive game, then the stable SB σ is nonempty-valued. Moreover, the stable value is given by: $E(\sigma(N))=(\mu(N)/n,\mu(N)/n,\ldots,\mu(N)/n)$.

<u>Proof:</u> In view of the symmetry of the game, for all $S\subset N$, $\sigma(S)$ is symmetric, that is, $x\epsilon\sigma(S)$ if and only if every permutation of x belongs to σ. Thus, if $\sigma(S)\neq\emptyset$, then $E(\sigma(S))=(\mu(S)/|S|,\mu(S)/|S|,\ldots,\mu(S)/|S|)$. It is, therefore, left to show that $\sigma(N)\neq\emptyset$. The proof is by induction on N (in fact, it is a constructive proof). For $n=1$ the validity of the claim is evident. Assume its validity for all games with less than n players, $n\geq2$, and let (N,μ) be an n-person game. Denote by m the size of a coalition whose per-capita payoff is maximal, that is, $\mu(S)/|S|$ is maximized for $|S|=m$. By the induction hypothesis, for all $S\subset N$, $S\neq N$, $\sigma(S)\neq\emptyset$. Hence, $E^i(\sigma(S))=\mu(S)/|S|$ for all $i\epsilon S$. Now if $m=n$ then $(\mu(N)/n,\mu(N)/n,\ldots,\mu(N)/n)\epsilon\sigma(N)$. Otherwise, consider the (sub)game $(\hat{N},\hat{\mu})$ where $\hat{N}\equiv\{m+1,\ldots,n\}$, and $\hat{\mu}$ is the restriction of μ to subsets of $\hat{N}$. Let $\hat{\sigma}$ be the stable SB for $(N,\hat{\mu})$. By the induction hypothesis, $\hat{\sigma}(\hat{N})\neq\emptyset$. Let $\hat{x}\epsilon\hat{\sigma}(\hat{N})$, and define $x\epsilon R^n_+$ as follows: $x^i\equiv\mu(M)/m$ if $i\epsilon M\equiv\{1,2,\ldots,m\}$, and $x^i\equiv\hat{x}^i$ if $i\notin M$. By superadditivity, $\hat{N}\cap M=\emptyset$ implies that

$\Sigma_{i\epsilon N} x^i \leq \mu(N)$. Thus, there exists $y \epsilon v^*(N)$ with $y^i \geq x^i$ for all $i \epsilon N$. By the choice of m and x, we have that $y \epsilon \sigma(N)$. Thus, $\sigma(N) \neq \emptyset$.

Q.E.D.

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